

Jubilee: Forecasting Long-Term Growth of S&P 500 Index

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Abstract This tutorial shows how to use the jubilee package in R to forecast the 10-year and 20-year growth trajectory of S&P 500 Index. R programs are provided to run the regressions, make predictions, and plot the charts. Additionally, in order to capture the short-horizon volatility (less than 10 years), we derive a new quantity that shows the deviation between the realized index level and the forecast. Interestingly, this quantity is mean-reverting in the same frequencies as the commonly-known business cycles. We attempt to locate potential leading indicator(s) for the next bear market and recession. A few factors are explored, among which yield spread inversion is considered most important.

Introduction

In Lihn (2018), we enhanced the CAPE model (Campbell and Shiller (1988); Campbell and Shiller (1998)) and provided a regression framework called the “Jubilee-Tectonic Model” to forecast the long-term growth of S&P 500 (SPX) index. The R^2 of the regression is above 80%, indicating very high consistency with historical data, and therefore, explanatory power. The data and utilities are packaged into the jubilee package in R. This tutorial provides an introduction how to use the package and generate the forecasts.

The main objective of long-term forecast is to regress the 10-year and 20-year forward nominal log-returns of SPX: $r_{f10}^{nom}(t)$ and $r_{f20}^{nom}(t)$, from which we can derive the predicted log-index $X_{pred}(t)$ from the realized total-return log-index $X(t)$. The 10-year forecast is popular in asset management, since the time horizon is reasonable for long-term asset allocators, such as pensions and endowments. However, as we found in Lihn (2018), the 20-year period matches the natural frequency of the long-term mean reversion cycles. Thus the 20-year regression requires the least tectonic adjustments.

The forecast horizon is called the “look-forward period”, denoted as $\Delta T_f = 10$ or 20. We construct the 5-factor forecast model as follows:

$$r_{f,\Delta T_f}^{nom}(t) \sim \beta_0 + \beta_1 Y(t) + \beta_2 R(t) + \beta_3 \text{CPI}_{10}(t) + \beta_4 \text{CPI}_{20}(t) + \beta_5 \log \text{CAPE}_{\Delta T_f}^{adj}(t) + \varepsilon; \quad (1)$$

$\text{CPI}_{10}(t)$ and $\text{CPI}_{20}(t)$ are the 10-year and 20-year log-returns of CPI (FRED symbol: CPIAUCSL). The channel return $R(t)$ and channel deviation $Y(t)$ are two important quantities derived from the trend-following channel in Section 2.3 of Lihn (2018). $Y(t)$ is the difference between $X(t)$ and the trend-following channel moving average $\alpha(t)$, that is, $Y(t) = X(t) - \alpha(t)$. $Y(t)$ is mean-reverting with an approximate 40-year cycle (See Figure 1 below).

We briefly review the concept of the trend-following channel and mean-reversion decomposition here. However, the reader can skip this paragraph since the detail here is encapsulated in the R package, and will not be used explicitly in this paper. Given a channel look-back period $\Delta T_b = 45$ years, at time t , we apply the causal regression $X(\tau) \sim \alpha(t) + R(t)(\tau - t)$, where $\tau \in [t - \Delta T_b, t]$, to produce $\alpha(t)$ and $R(t)$, and therefore $Y(t)$. Using the discrete notation on $X(\tau)$ and τ , the input data points to the regression are $\{(X_i = X(\tau_i), \tau_i), \forall i = 1 \cdots N, \tau_i \in [t - \Delta T_b, t]\}$. And the regression yields analytical solutions: $\alpha(t) = \langle X_i \rangle + \sqrt{3 \left(\frac{N+1}{N} \right) \text{Cor}(X_i, \tau_i) \text{Stdev}(X_i)}$; $R(t) = \text{Cor}(X_i, \tau_i) \frac{\text{Stdev}(X_i)}{\text{Stdev}(\tau_i)}$; and $Y(t) = X(t) - \langle X_i \rangle - \sqrt{3 \left(\frac{N+1}{N} \right) \text{Cor}(X_i, \tau_i) \text{Stdev}(X_i)}$. From the $\langle X_i \rangle$ term in $\alpha(t)$, we can see why $\alpha(t)$ resembles some kind of moving average.

We assume the in-sample window is $t \in [T_{start}, T_{end}]$ during which period all the factors are available. Due to the forward-looking nature of $r_{f,\Delta T_f}^{nom}(t)$, it is NaN after $t > T_{end} - \Delta T_f$. Thus the regression is performed for $t \in [T_{start}, T_{end} - \Delta T_f]$, and the factor loadings are used to make prediction, denoted as $\left[r_{f,\Delta T_f}^{nom}(t) \right]_{pred}$, that extends to $t \in [T_{end} - \Delta T_f, T_{end}]$.

The log-CAPE is tectonically adjusted by the fault lines $\left\{ \left(t_i^{adj}, \Delta_i \log \text{CAPE}_{\Delta T_f} \right), \forall i = 1 \cdots N \right\}$ with the linear “ramp-up” period Δt_{ramp} such that

$$\log \text{CAPE}_{\Delta T_f}^{\text{adj}}(t) = \log \text{CAPE}_{\Delta T_f}(t) + \sum_{i=1 \dots N} \begin{cases} 0, & t < t_i^{\text{adj}}; \\ \frac{t - t_i^{\text{adj}}}{\Delta t_{\text{ramp}}} \Delta_i \log \text{CAPE}_{\Delta T_f} & t \in [t_i^{\text{adj}}, t_i^{\text{adj}} + \Delta t_{\text{ramp}}]; \\ \Delta_i \log \text{CAPE}_{\Delta T_f}, & t \geq t_i^{\text{adj}} + \Delta t_{\text{ramp}}, \end{cases} \quad (2)$$

where $\text{CAPE}_{\Delta T_f}$ is the cyclically adjusted P/E ratio for ΔT_f years. Each tuple $(t_i^{\text{adj}}, \Delta_i \log \text{CAPE}_{\Delta T_f})$ describes a jump in log-CAPE at time t_i^{adj} , which is called a “fault line”, as if the economy was going through an earth quake and a crack was left on the ground.

When $\Delta t_{\text{ramp}} \rightarrow 0$, the fault line adjustments converge to the step function approach described in Section 1.2 of [Lihn \(2018\)](#). The purpose of introducing the Δt_{ramp} parameter is to eliminate the discontinuity in the index prediction $X_{\text{pred}}(t)$ such that the short-horizon mean-reversion quantity $Z_{\Delta T_f}(t)$ in Eq. (4) makes more sense. We apply $\Delta t_{\text{ramp}} = 5$ to the 20-year forecast in this tutorial. It is set to zero for the 10-year forecast so that it is consistent with [Lihn \(2018\)](#).

The objective of optimization is to minimize the AIC of the regression by a nonlinear optimization on the fault lines as well as the factor loadings $\{\beta_i\}$. We view our tectonic 5-factor model as a meaningful extension from the one-factor CAPE model, $r_{f, \Delta T_f}^{\text{real}}(t) \sim \log \text{CAPE}_{\Delta T_f}(t) + \varepsilon$.

The nominal return forecast can be translated into index level prediction without the complication of inflation forecast. The regressed forward return $r_{f, \Delta T_f}^{\text{nom}}(t)$ is converted to the predicted log-index $X_{\text{pred}}(t)$ and future SPX level (dividend included) $p_{\text{pred}}(t)$ via (See Section 7 of [Lihn \(2018\)](#))

$$\begin{aligned} X_{\text{pred}}(t + \Delta T_f) &= X(t) + \left[r_{f, \Delta T_f}^{\text{nom}}(t) \right]_{\text{pred}} \Delta T_f, \text{ where } t \in [T_{\text{start}}, T_{\text{end}}]; \\ \text{and } p_{\text{pred}}(t) &= p(T_{\text{end}}) \exp(X_{\text{pred}}(t) - X(T_{\text{end}})). \end{aligned} \quad (3)$$

The the total-return price $p(t)$ is rebased to the price level of the most recent month, using the relation between $p(T_{\text{end}})$ and $X(T_{\text{end}})$, such that $p(t) = p(T_{\text{end}}) \exp(X(t) - X(T_{\text{end}}))$. Therefore, the predicted total-return price $p_{\text{pred}}(t)$ is calculated in the same manner.

The difference between the realized log-index path $X(t)$ and the predicted path $X_{\text{pred}}(t)$ yields a new quantity, called the “**short-horizon mean-reversion**” (SMR) index:

$$\begin{aligned} Z_{\Delta T_f}(t) &\equiv X(t) - X_{\text{pred}}(t) \Big|_{\Delta T_f}, \text{ where } t \in [T_{\text{start}}, T_{\text{end}}]; \\ &= X(t) - X(t - \Delta T_f) - \left[r_{f, \Delta T_f}^{\text{nom}}(t - \Delta T_f) \right]_{\text{pred}} \Delta T_f. \end{aligned} \quad (4)$$

We find that $Z_{\Delta T_f}(t)$ is mean-reverting between -0.5 and 0.5 (see [Figure 7](#)), in the same frequencies as the commonly-known business cycles, about 5-10 years. Thus it is called the “short horizon”.

From the perspective of $Y(t)$, we have the alternative expression:

$$Z_{\Delta T_f}(t) \equiv Y(t) - Y_{\text{pred}}(t) \Big|_{\Delta T_f}, \text{ where } Y_{\text{pred}}(t) \equiv X_{\text{pred}}(t) - \alpha(t). \quad (5)$$

That is, $Z_{\Delta T_f}(t)$ is the residual between the realized $Y(t)$ and the predicted $Y(t)$, thus mean-reverting in much shorter cycles.

This quantity explains the difference between long-term investing and short-term market-timing. For long-term investors, $Z_{\Delta T_f}(t)$ can be considered as “short-term noises.” But for savvy portfolio managers, if its behavior is predictable, its pattern can be taken advantage of in each business cycle. We find that $Z_{\Delta T_f}(t)$ is deeply correlated to the interest rate policy, and more specifically, the yield curve inversion. It will be briefly touched upon in this tutorial.

Loading Package and Preparing Data

We begin with loading the jubilee package and setting up several essential data tables:

```
> library(jubilee)
> repo <- jubilee.repo(online=FALSE)

[1] "Maximum date in ie.data is 2019.07 and SPX average at 2975.95"
[1] "Maximum date for unrte is 2019-06-16 and for GDP, 2019-02-15"
```

```

> ju <- jubilee(repo@ie, lookback.channel=45, fwd.rtn.duration=10)
> dt <- ju@dtb
> dj <- ju@reg.dtb

```

The `repo` object is an instance of the `jubilee.repo` class. It stores the raw data, mainly in the `repo@ie` data table slot, which is named after “irrational exuberance” to honor the main data source of [Shiller \(2018\)](#). The `ju` object is an instance of the `jubilee` class that most functions are based on. Inside the `ju` object, the `dt=ju@dtb` data table contains simple time series, many are directly copied from the `repo@ie` data table. The `dj=ju@reg.dtb` data table contains derived time series required for regression. The separation is to prevent each data table from getting too bulky. We will work with the `dj` data table most of the time.

The `lookback.channel` parameter is set to 45 years, according to Section 2.4 of [Lihn \(2018\)](#). The user is not expected to change this setting. The `fwd.rtn.duration` parameter corresponds to ΔT_f , which is set to 10 for the 10-year forecast.

The package comes with static data going back 100-200 years. The static data will be only updated by the infrequent releases of the package. The user can choose to amend the most recent data from the internet by specifying `online=TRUE`. This will allow you to make timely forecasts.

The mapping between the mathematical notations and the columns in the data tables is described as follows:

- $t = dj\$fraction$: Time in years. Each month is in the $1/12$ unit. Note that we follow Shiller’s “middle of the month” convention since he averages the quantity. But when we download the monthly data from FRED, we use the monthly data as is.
- $X(t) = dj\$log.tri$: The logarithm of the total return index for SPX.
- $r_{f10}^{nom}(t) = dj\$eqty.logr.f10$: The 10-year forward returns of $X(t)$.
- $r_{f20}^{nom}(t) = dj\$eqty.logr.f20$: The 20-year forward returns of $X(t)$.
- $CPI_{10}(t) = dj\$cpi.logr.10$: The 10-year (backward) log-return of CPI.
- $CPI_{20}(t) = dj\$cpi.logr.20$: The 20-year (backward) log-return of CPI.
- $R(t) = dj\$eqty.lm.r$: The channel return.
- $Y(t) = dj\$eqty.lm.y$: The channel deviation.
- $\alpha(t) = dj\$eqty.lm.a$: The trend-following channel moving average.
- $\log(CAPE_{10}(t)) = dj\$log.cape.10$: The 10-year log-CAPE.
- $\log(CAPE_{20}(t)) = dj\$log.cape.20$: The 20-year log-CAPE.

As a starter, the following R code demonstrates a plot on the channel deviation $Y(t)$ via the corresponding column `dj$eqty.lm.y` in the data table. $Y(t)$ is the major analytical quantity that enhances the CAPE model. In [Figure 1](#), we observe four bottoms (1896, 1932, 1974, 2009) approximately 40 years apart. Such long cycle is essential for the long-term forecast model. But it is too long to work with for the business cycle forecast, which we will address in Section “[Short-Horizon Mean Reversion](#)”.

```

> J <- which(dj$fraction > 1881) # defines the in-sample window
> plot(dj$fraction[J], dj$eqty.lm.y[J], col="blue", type="l",
+      xlab="$t$", ylab="$Y(t)$",
+      main="The channel deviation Y(t)")
> abline(h=0, col="red", lty=2)

```

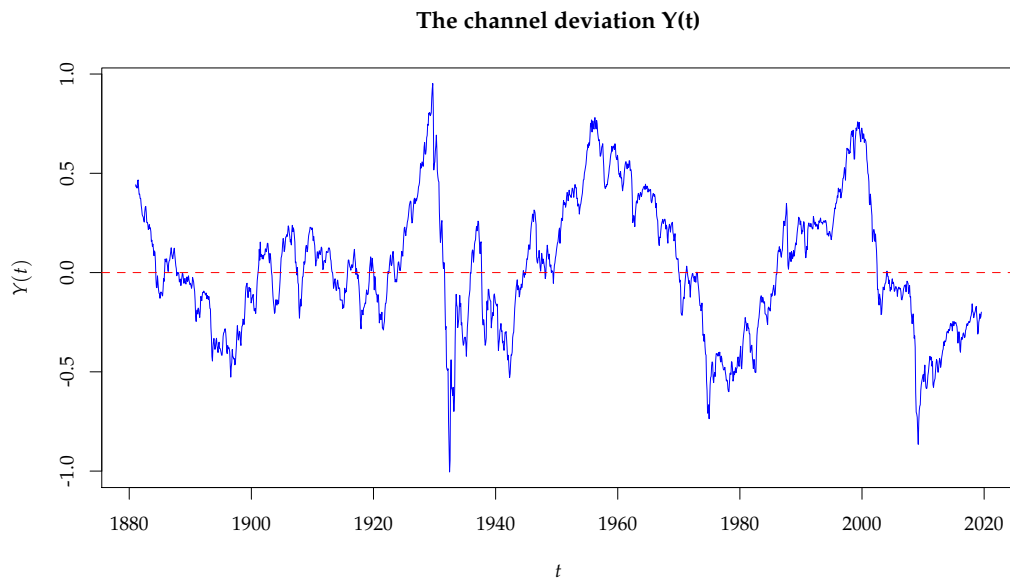


Figure 1: The channel deviation $Y(t)$. It is mean-reverting with an approximate 40-year cycle.

Regression of The 10-Year Returns

The fault lines $\left\{ \left(t_i^{adj}, \Delta_i \log \text{CAPE}_{\Delta T_f} \right), \forall i = 1 \dots N \right\}$ have been pre-packaged into data sets inside the function `jubilee.std_fault_line()`. The user only needs to choose which data set to use. Here we use “`r_nom_f10_5ftr_4f1`”, which means “for nominal return regression, forward 10 years, 5-factors, 4 fault lines”. Its content is shown below:

```

> fl.10 <- jubilee.std_fault_line("r_nom_f10_5ftr_4f1")
> fl.10

  fraction shift
1 1907.06  1.464
2 1935.61 -1.499
3 1944.48 -1.245
4 1985.85 -0.511

```

The following R code shows the procedure described in Eq. (Introduction) and Eq. (Introduction) to adjust log-CAPE with the fault lines, perform the regression, and generate the prediction:

```

> dj$log.cape10.adj <- jubilee.adj_fault_line(dj$fraction, dj$log.cape10, fl.10)
> J <- which(dj$fraction > 1881) # defines the in-sample window
> df <- dj[J,] # defines the in-sample data set
> lm.10 <- lm(eqty.logr.f10 ~
+           eqty.lm.y + eqty.lm.r + log.cape10.adj + cpi.logr.10 + cpi.logr.20,
+           data=df)
> pred.10 <- jubilee.predict(ju, lm.10, df)
> summary(lm.10)

```

Call:

```

lm(formula = eqty.logr.f10 ~ eqty.lm.y + eqty.lm.r + log.cape10.adj +
    cpi.logr.10 + cpi.logr.20, data = df)

```

```

Residuals:
      Min       1Q   Median       3Q      Max
-0.109464 -0.012505 -0.000042  0.013734  0.066082

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.4059518  0.0064600  62.840  <2e-16 ***
eqty.lm.y     -0.0885792  0.0018780 -47.167  <2e-16 ***
eqty.lm.r     -2.7815494  0.0596564 -46.626  <2e-16 ***
log.cape10.adj -0.0432022  0.0008104 -53.310  <2e-16 ***
cpi.logr.10   -0.0576107  0.0341383  -1.688  0.0917 .
cpi.logr.20    0.9964918  0.0442087  22.541  <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.02042 on 1536 degrees of freedom
(121 observations deleted due to missingness)
Multiple R-squared:  0.8135,    Adjusted R-squared:  0.8129
F-statistic: 1340 on 5 and 1536 DF,  p-value: < 2.2e-16

```

The summary statistics shows R^2 above 80%, indicating the regression has very high explanatory power. The prediction logic is encapsulated in the `jubilee.predict()` function, which produces the `pred.10` data table, containing many useful columns in a single table. The some column definitions in `pred.10` are described below:

- $X_{pred}(t) = \text{pred.10}\$log.tri$: The predicted log-index.
- $\left[r_{f10}^{nom}(t) \right]_{pred} = \text{pred.10}\$logr$: The predicted nominal returns.
- $p_{pred}(t) = \text{pred.10}\$price$: The predicted SPX total-return price, rebased according to the SPX price of the most recent month.
- $Y_{pred}(t) = \text{pred.10}\$eqty.lm.y$: The predicted $Y(t)$.
- $X(t) = \text{pred.10}\$log.tri.source$: The realized log-index. (An interpolated copy.)
- $p(t) = \text{pred.10}\$tri.source$: The realized SPX total-return price, rebased according to the SPX price of the most recent month. (An interpolated copy.)
- $\alpha(t) = \text{pred.10}\$eqty.lm.a.source$: The realized $\alpha(t)$. (An interpolated copy.)

The forward-return regression can be visualized in Figure 2 by the following R code:

```

> plot(pred.10$fraction[J], pred.10$logr[J]*100, col="red", type="l",
+       xlab="$t$", ylab="$r_{f10}^{nom}(t)$",
+       main="10-year nominal forecast, 5-factor tectonic CAPE model")
> lines(df$fraction, df$eqty.logr.f10*100, col="blue")
> for (i in 1:NROW(fl.10)) { abline(v=fl.10[i,1], col="blue", lty=2) } # faults

```

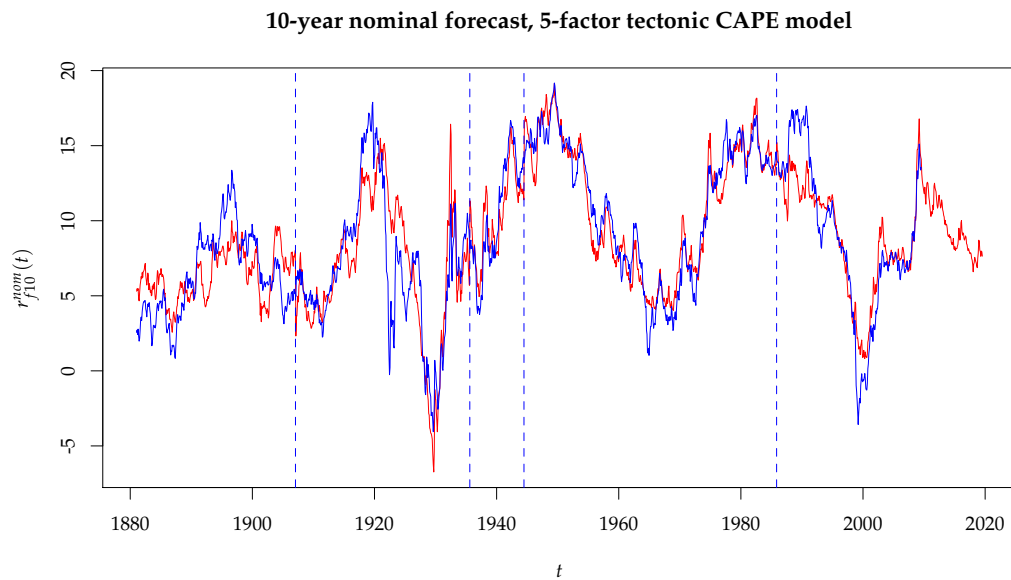


Figure 2: The 10-year nominal return forecast by 5-factor tectonic CAPE model. The blue line is the realized returns. The red line is the predicted returns. The dotted blue lines are the locations of the faults.

We can also visualize the log-index as well as the price-level forecast since 1997 via the following plots in Figure 3. We observe that the predicted index is much smoother than the actual index. The prediction is of the long-term nature, forecasting SPX will exceed 5000 in ten years (dividend included). But it leaves out the short-horizon fluctuations unexplained, such as the late-1990 rally, the dot-com crash, and the 2008-2009 crash and rebound.

```

> J10 <- which(pred.10$fraction >= 1997)
> par(mfcol=c(2,1))
> plot(pred.10$fraction[J10], pred.10$log.tri[J10], col="red", type="l",
+       xlab="$t$", ylab="$X(t)$",
+       main="Forecast of log-index X(t) in 10y model")
> lines(dj$fraction, dj$log.tri, cex=0.1, col="blue")
>
> plot(pred.10$fraction[J10], pred.10$price[J10], col="red", type="l",
+       xlab="$t$", ylab="$p(t)$",
+       main="Forecast of price index p(t) in 10y Model")
> lines(pred.10$fraction, pred.10$tri.source, cex=0.1, col="blue")
> abline(h=3000, col="black", lty=3)

```

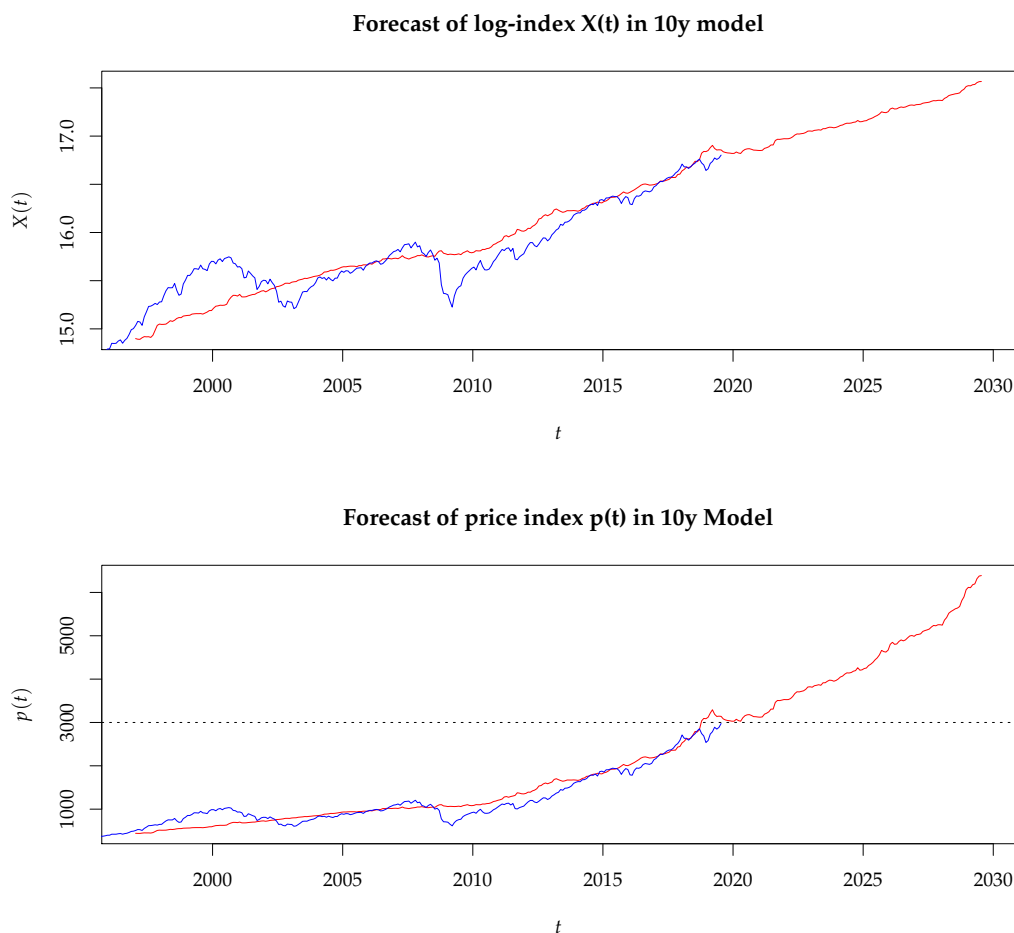


Figure 3: The log-index $X(t)$ and the price-level $p(t)$ forecasts since 1997 in the 10-year model. The blue lines are the realized values. The red lines are the predicted values.

Forecast in The 20-Year Model

It was mentioned in the introduction that the 20-year period matches the natural frequency of the long-term mean-reversion cycles. Thus the 20-year regression requires the least tectonic adjustments. We also find that the 20-year model is more suitable to describe the short-horizon fluctuation. This gives us the incentive to explore it in depth here in the tutorial.

We repeat the data loading procedure, except that the fwd.rtn.duration parameter ΔT_f is set to 20 years:

```

> repo <- jubilee.repo(online=FALSE)

[1] "Maximum date in ie.data is 2019.07 and SPX average at 2975.95"

```

```
[1] "Maximum date for unrte is 2019-06-16 and for GDP, 2019-02-15"
> ju <- jubilee(repo@ie, lookback.channel=45, fwd.rtn.duration=20)
> dt <- ju@dtb
> dj <- ju@reg.dtb
```

Next, we use the fault line data set "r_nom_f20_5ftr_2fl_ramp5y", which means "for nominal return regression, forward 20 years, 5-factors, 2 fault lines, 5-year ramp-up", shown below. The 5-year ramp-up time smooths out the two jumps caused by the fault lines.

```
> fl.20 <- jubilee.std_fault_line("r_nom_f20_5ftr_2fl_ramp5y")
> fl.20
```

```
fraction shift
1 1902.42 -1.077
2 1929.20 -1.499
```

Note that the two fault lines are before WWII. We like the fact that they are not intrusive to the modern history after WWII. This indicates that, from the perspective of 20-year window, the modern economic history has been quite stable. The forward-return regression is performed as following:

```
> dj$log.cape20.adj <- jubilee.adj_fault_line(dj$fraction, dj$log.cape20, fl.20, months=12*5)
> J <- which(dj$fraction > 1881)
> df <- dj[J,]
> lm.20 <- lm(eqty.logr.f20 ~
+           eqty.lm.y + eqty.lm.r + log.cape20.adj + cpi.logr.10 + cpi.logr.20,
+           data=df)
> pred.20 <- jubilee.predict(ju, lm.20, df)
> summary(lm.20)
```

Call:

```
lm(formula = eqty.logr.f20 ~ eqty.lm.y + eqty.lm.r + log.cape20.adj +
    cpi.logr.10 + cpi.logr.20, data = df)
```

Residuals:

```
      Min       1Q   Median       3Q      Max
-0.042841 -0.007359  0.000219  0.006398  0.046093
```

Coefficients:

```
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.0951992  0.0020374   46.73 <2e-16 ***
eqty.lm.y     -0.0262312  0.0011001  -23.84 <2e-16 ***
eqty.lm.r      0.3889991  0.0248760   15.64 <2e-16 ***
log.cape20.adj -0.0248460  0.0003953  -62.86 <2e-16 ***
cpi.logr.10    0.3253106  0.0165312   19.68 <2e-16 ***
cpi.logr.20   -0.8882144  0.0264468  -33.59 <2e-16 ***
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Residual standard error: 0.011 on 1297 degrees of freedom

(360 observations deleted due to missingness)

Multiple R-squared: 0.8617, Adjusted R-squared: 0.8611

F-statistic: 1616 on 5 and 1297 DF, p-value: < 2.2e-16

The summary statistics shows R^2 above 85%, indicating the regression has very high explanatory power. All the factors are highly significant. The return regression can be visualized in Figure 4 by the following R code:


```

> plot(pred.20$fraction[J], pred.20$logr[J]*100, col="red", type="l",
+       xlab="$t$", ylab="$r_{f20}^{nom}(t)$", ylim=c(2,17),
+       main="20-year nominal forecast, 5-factor tectonic CAPE model")
> lines(df$fraction, df$eqty.logr.f20*100, col="blue")
> for (i in 1:NROW(fl.20)) { abline(v=fl.20[i,1], col="blue", lty=2) } # faults

```

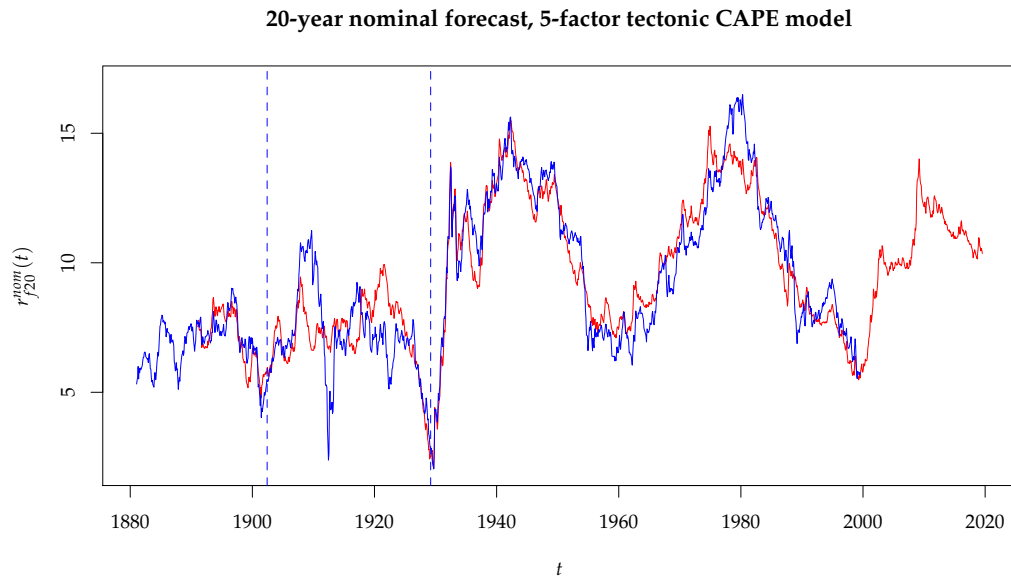


Figure 4: The 20-year nominal return forecast by 5-factor tectonic CAPE model. The blue line is the realized returns. The red line is the predicted returns. The dotted blue lines are the locations of the faults.

Prediction of $X(t)$ and $Y(t)$

The log-index forecast from the 20-year model is shown in the upper panel of Figure 5. The blue line is the realized $X(t)$. The solid red line is the predicted path from the 20-year model. The dotted red line is the predicted path from the 10-year model. We observe that the two red lines generally agree. But the 20-year model produces slightly more optimistic forecast than the 10-year model for the next decade. Such optimism looks more pronounced in the price level forecast in the lower panel.

```

> par(mfcol=c(2,1))
> J20 <- which(pred.20$fraction >= 1988 & pred.20$fraction <= 2038)
> plot(pred.20$fraction[J20], pred.20$log.tri[J20], type="l", col="red",
+       xlab="$t$", ylab="$X(t)$",
+       main="Prediction of log index X(t) in 20y model")
> lines(dt$fraction, dt$log.tri, cex=0.1, col="blue")
> lines(pred.10$fraction[J20], pred.10$log.tri[J20], type="l", col="red", lty=2)
>
> J20a <- which(pred.20$fraction >= 1995 & pred.20$fraction <= 2025)
> plot(pred.20$fraction[J20a], pred.20$price[J20a], col="red", type="l",
+       xlab="$t$", ylab="$p(t)$",
+       main="Forecast of price index p(t) in 20y Model")
> lines(pred.10$fraction, pred.10$tri.source, cex=0.1, col="blue")
> lines(pred.10$fraction, pred.10$price, type="l", col="red", lty=2)
> abline(h=3000, col="black", lty=3)
    
```

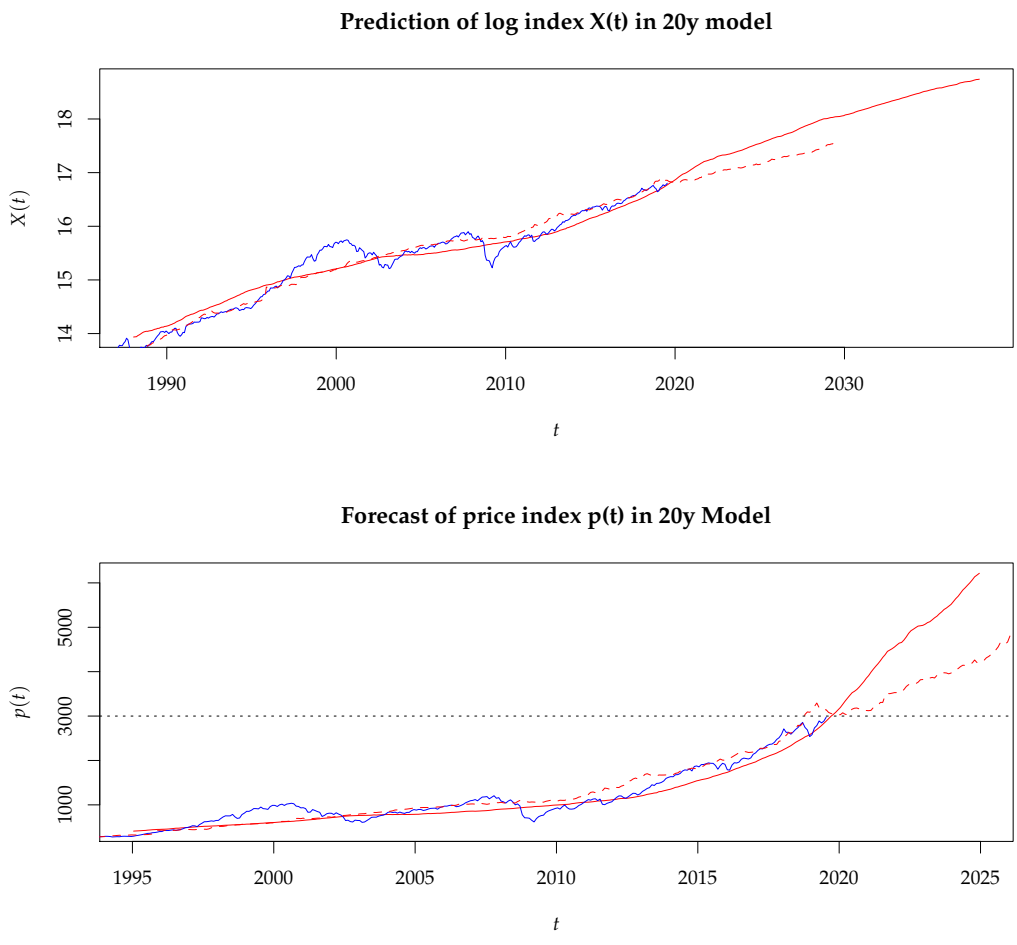


Figure 5: Upper Panel: The log-index $X(t)$ forecasts since 1988 in both the 10-year and 20-year models. The blue line is the realized $X(t)$. The solid red line is the predicted path from the 20-year model. The dotted red line is the predicted path from the 10-year model. **Lower Panel:** The price-level $p(t)$ forecasts in both the 10-year and 20-year models.

We notice again that the predicted path is much smoother than the realized path in the in-sample period. This difference can be best illustrated by the $Y(t)$ plot in Figure 6. The blue line is the realized $Y(t)$. The solid red line is $Y_{pred}(t)$ from the 20-year model. Their difference leads us to define the next important quantity: the “**short-horizon mean reversion**” index, which captures the impact from the business cycles. It will be explored in the rest of this paper.

```

> par(mfcol=c(1,1))
>
> JY <- which(dj$fraction > 1912)
> plot(dj$fraction[JY], dj$eqty.lm.y[JY], type="l", col="blue",
+     xlab="$t$", ylab="$Y(t)$",
+     main="Predicted and realized Y(t) in 20y model")
> lines(pred.20$fraction[JY], pred.20$eqty.lm.y[JY], col="red")
> lines(pred.10$fraction[JY], pred.10$eqty.lm.y[JY], col="red", lty=2)
> rect(1937, 0.5, 1945, 0.7, col="orange") # WWII

```

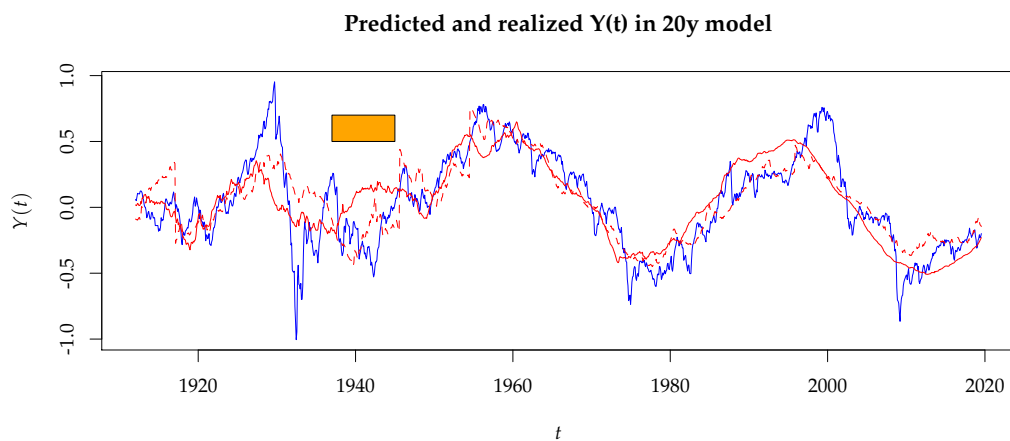


Figure 6: The blue line is the realized $Y(t)$, since 1912. Since $Y(t)$ is mean-reverting, a clear historical plot can be presented. The solid red line is the predicted path $Y_{pred}(t)$ from the 20-year model. The predicted path is much smoother than the realized path. As a comparison, the dotted red line is $Y_{pred}(t)$ from the 10-year model. The orange box represents WWII.

Short-Horizon Mean Reversion

The long-term model leaves ample room for short-horizon fluctuations. If you are truly a buy-and-hold long-term investor, this fluctuation should not trouble you. However, individuals and organizations have utilities in limited time frame. Asset bubble forces you to buy at expensive prices. Steep downside fluctuation becomes an acute problem when you have to withdraw funds from your investment. In this section, we provide a preliminary framework to understand such fluctuation. This topic is still under research, thus the exploration here is quite open-ended.

An interesting outcome of the 20-year model is that the “**short-horizon mean reversion**” (SMR) index is least noisy and most meaningful. It is mainly due to the fact that there is no fault line adjustment after WWII.

$$\begin{aligned} Z_{20}(t) &\equiv X(t) - X_{pred}(t) \Big|_{\Delta T_f=20} \\ &\equiv Y(t) - Y_{pred}(t) \Big|_{\Delta T_f=20} \end{aligned} \quad (6)$$

$Z_{20}(t)$ is stored in the `log.tri.smr` column in the `pred.20` data table. Its style statistics is shown below:

```
> mean(pred.20$log.tri.smr, na.rm=TRUE)
[1] -1.456836e-05
> sd(pred.20$log.tri.smr, na.rm=TRUE)
[1] 0.2197144
> moments::skewness(pred.20$log.tri.smr, na.rm=TRUE)
[1] 0.1614918
> moments::kurtosis(pred.20$log.tri.smr, na.rm=TRUE)
[1] 4.305534
```

As one can see, its mean is zero, thus a mean-reverting process. As shown in Figure 7, whenever $Z_{20}(t)$ swings to 2-stdev, it usually indicates the market is in extreme conditions, either over-sold or over-bought.

```

> J <- which(dj$fraction > 1912)
> plot(pred.20$fraction[J], pred.20$log.tri.smr[J], type="l", col="blue",
+       ylab="$Z_{20}(t)$", xlab="$t$",
+       main="Short-horizon mean reversion in 20y model")
> abline(h=0, col="red", lty=2)
> sd2 <- 2*sd(pred.20$log.tri.smr, na.rm=TRUE)
> abline(h=sd2, col="blue", lty=2)
> abline(h=-sd2, col="blue", lty=2)
> rect(1937, 0.5, 1945, 0.7, col="orange") # WWII

```

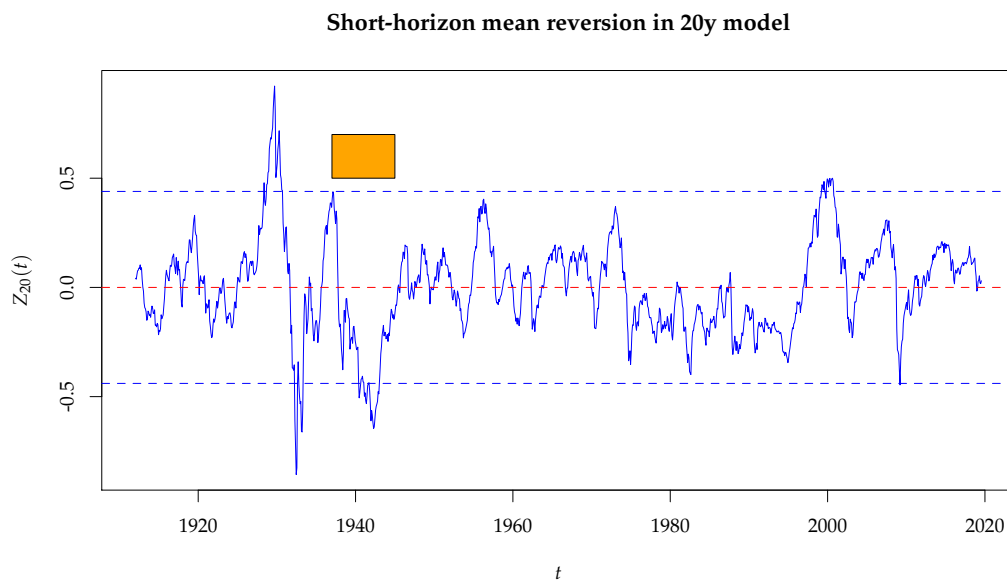


Figure 7: The short-horizon mean reversion (SMR) index $Z_{20}(t)$, drawn in the blue line. The dotted blue lines are the ± 2 -stdev of $Z_{20}(t)$. The orange box represents WWII.

The log-index $X(t)$ can be viewed as a decomposition of the long-term component $X_{pred}(t)$ and the short-term component $Z_{20}(t)$. The long-term component can be predicted by our enhanced "Tectonic CAPE" model. What drives the short-term component? Our view is that $Z_{20}(t)$ is mainly influenced by the interest rate policy, barring major world wars. We will also examine the unemployment rate and inflation.

Interest Rate Policy and Yield Spread

Yield curve inversion has been found to be one of the most reliable leading indicators of the past several large-scale bear markets. Yield spread is the most powerful policy tool that affects all aspects of macro-economics.

The economic reasoning of yield curve inversion is quite appealing: The stock market can be viewed as a large financial operation that borrows on the short-term rate and lends on the long-term rate. Companies such as banks can build leverage to enhance their returns from the spread. But some companies can use excessive leverage to "game the system", so to speak. Thus, modulating the yield spread is a key policy tool for FED to tame the speculation. When the yield spread turns negative, there is no profit to be made from the spread, causing dismal earnings for several quarters, and the stock price crashes due to lower valuation. A few of them with poor balance sheet may go into bankruptcy. This chilling effect usually lingers for about 12-24 months.

Furthermore, the narrowing of yield spread creates a regime switching scenario. In a red hot economy, as the yield spread becomes smaller, some executives would not think of reducing their balance sheet risk. Instead, many of them would choose to increase the leverage, in order to generate equal or more profit and justify the rising stock price. Suddenly, the yield curve is inverted, the mathematics of leveraging reaches a singularity. And the balance sheets of these companies collapse.

Later, we will introduce the concept of "unemployment rate as high-yield bond". This extends the influence of interest rate policy far beyond the financial sector. In the internet age, the largest companies

are all in the consumer sector selling goods and products to the vast consumers. Employment rate is their borrowing.

In some occasions, FED's tightening was in a straight path to inversion. But in other times, FED might swing in short cycles of tightening and easing before touching down on the ultimate inversion. The current trend points to next inversion around year 2020, but it is totally up to the FED to make it happen or not.

Given the following attributes:

- $GS_{10}(t) = \text{dj\$rate.gs10}$: the yield of 10-year Treasury,
- $GS_{3m}(t) = \text{dj\$rate.tb3ms}$: the yield of 3-month Treasury bill,
- $YS(t) \equiv GS_{10}(t) - GS_{3m}(t) = \text{dj\$rate.spread}$: the yield spread,

we define the centered yield spread normalized by the 10-year yield:

$$\widehat{YS}(t) \equiv \frac{YS(t) - \langle YS(t) \rangle}{GS_{10}(t)}. \quad (7)$$

It is calculated and stored in `dj$rate.spread.norm` column. Here $\langle YS(t) \rangle$ is the mean of $YS(t)$, which is approximately 1.5% as shown by the variable `rate.spread.mean` below, which is also a slot in the `ju` object. It is the equilibrium level of yield spread.

Our first finding is that $Z_{20}(t)$ is deeply related to $\widehat{YS}(t)$. This is illustrated in Figure 8. The two quantities have nearly the same magnitude, that is, $Z_{20}(t) \approx \widehat{YS}(t)$. They often move in tandem. Every yield curve inversion (shown as vertical dotted red lines) coincides with the market top, and the following easing matches the bottom of the selloff. For the equity market, yield spread inversion is the decisive turning point that causes regime switching from a bull market to a bear market. On the contrary, the market switches back from a bear market to a bull market expeditiously once the yield spread is widened large enough.

```

> rate.spread.mean <- mean(dj$rate.spread, na.rm=TRUE)
> rate.spread.mean

[1] 1.459764

> J <- which(dj$fraction > 1912)
> ma <- function(x,n,sides=1){filter(x,rep(1/n,n), sides=sides)}
> plot(pred.20$fraction[J], pred.20$log.tri.smr[J], type="l", lwd=1, col="blue",
+       ylab="$Z_{20}(t)$ vs $\widehat{YS}(t)$", xlab="$t$",
+       main="Short-horizon mean reversion and yield spread inversion")
> lines(dj$fraction[J], ma(-dj$rate.spread.norm,3)[J], type="l", col="red")
> for (t in repo@yield.inversion) { abline(v=t, lty=2, col="red") }
> abline(h=0, col="blue", lty=2)
> rect(1937, 0.5, 1945, 0.7, col="orange") # WWII

```

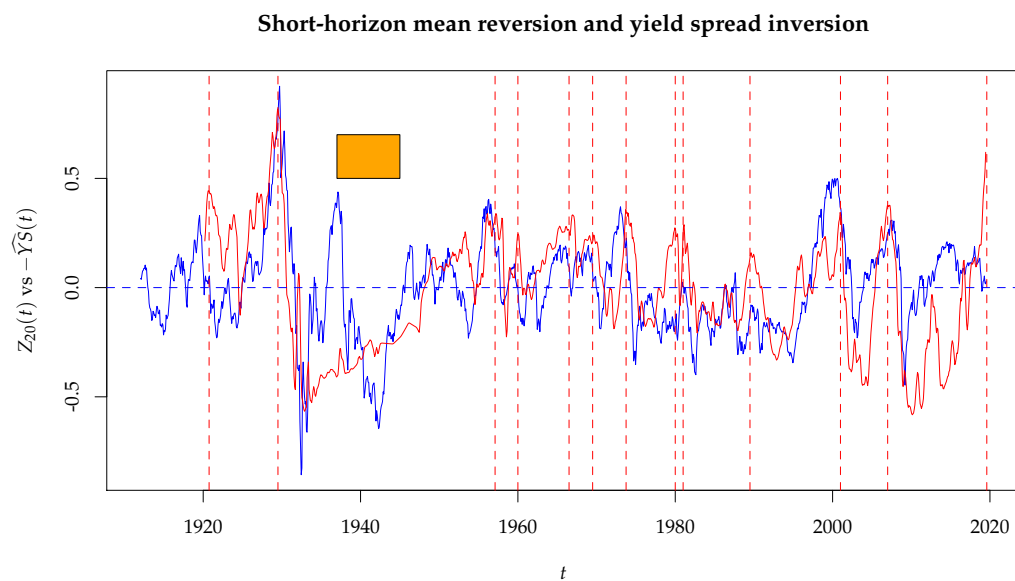


Figure 8: Overlay of normalized yield spread $-\widehat{YS}(t)$ (red line) with $Z_{20}(t)$ (blue line). The vertical dotted red lines are the locations of maximum yield spread inversion in each business cycle. Here $\widehat{YS}(t)$ is smoothed by 3-month moving average to make it less noisy in the plot. The orange box represents WWII.

The Uncertainty Principle

How wide of the yield spread is “wide enough”? Is the fluctuation of the stock market affected by the interest rate environment it is in? We think this is related to the following quantity. Define the modified inverse of 10-year yield $GSI(t)$ as

$$GSI(t) \equiv \frac{\langle YS(t) \rangle}{GS_{10}(t)}. \quad (8)$$

It is calculated and stored in `dj$rate.gs10.modinv` column. The fact that $Z_{20}(t) \approx \widehat{YS}(t)$ implies that the maximum amplitude of $Z_{20}(t)$ is approximately $GSI(t)$, which associates the volatility of the equity market with the inverse of the long-term interest rate. We call this relation the **Uncertainty Principle**. This explains that, on one hand, in a low interest rate environment, the equity market is susceptible to high volatility. On the other hand, the equity market is fairly dormant in a high interest rate environment, e.g. Dow was flat from 1966 to 1981.

Figure 9 adds the overlay of $GSI(t + 74/12)$ with $Z_{20}(t)$ and $\widehat{YS}(t)$. The 74-month shift is described in Section 9.1 of Lihn (2018) as the equity market’s lead time over GS_{10} . We draw $\pm GSI(t)$ in purple lines to form an envelope, and we smooth $GSI(t)$ by its 24-month moving average to remove the noise.

We observe that the $GSI(t)$ envelope indeed describes the fluctuation of $Z_{20}(t)$ quite well. The

equity market went through very large swings whenever the Treasury yield was trending into the low interest rate period. The first incident was marked by the big rally from 1921 to the 1929 peak, followed by a decade of turmoils, including WWII. The second incident was marked by the irrational exuberance leading to the 2000 peak, followed by two severe bear markets in the next decade.

Statistically speaking, it is far from certain to prove whether such statement can be true or not. The GS10 cycle is so long (80 years) that you won't see the next low interest rate environment until the end of the century.

```
> plot(pred.20$fraction[J], pred.20$log.tri.smr[J], type="l", lwd=1, col="blue",
+       ylab="$Z_{20}(t)$", xlab="$t$",
+       main="Uncertainty principle: Z(t) and inverse of GS10 (GSI)")
> rect(1937, 0.8, 1945, 0.95, col="orange") # WWII
> lines(dj$fraction[J], ma(-dj$rate.spread.norm,3)[J], type="l", col="red")
> abline(h=0, col="blue", lty=2)
> GSI <- dj$rate.gs10.modinv
> lines(dj$fraction[J]-74/12, ma(GSI,24)[J], col="purple", lty=2, lwd=2)
> lines(dj$fraction[J]-74/12, -ma(GSI,24)[J], col="purple", lty=2, lwd=2)
> lines(dj$fraction[J], GSI[J], col="purple", lty=3, lwd=1)
```

Uncertainty principle: Z(t) and inverse of GS10 (GSI)

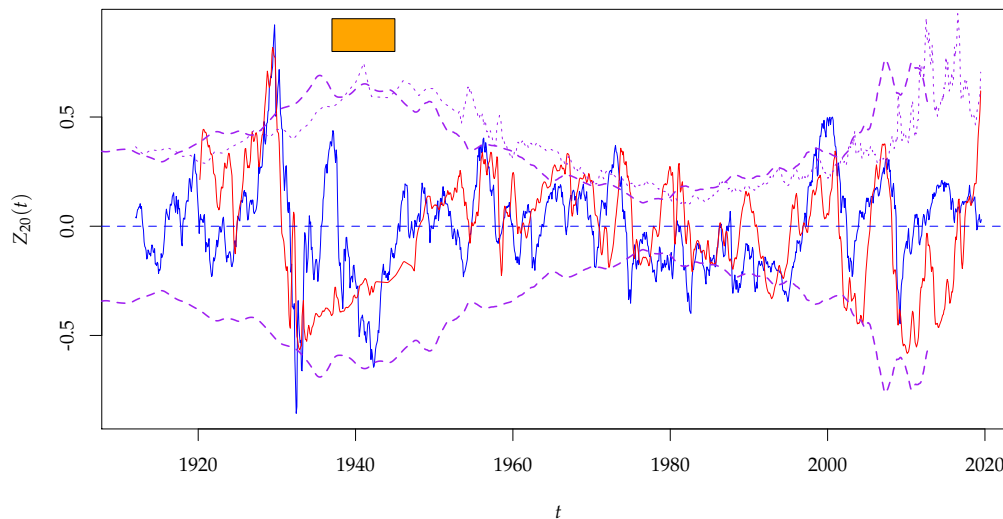


Figure 9: Overlay of $GSI(t + 74/12)$ with $Z_{20}(t)$ (the blue line). The 74-month shift is the equity market's lead time over Treasury market. We draw $\pm GSI(t + 74/12)$ in thick purple dotted lines to form an envelope, which are smoothed by 24-month moving average to remove the noise. The red line is the yield spread. We draw $GSI(t)$ in thin purple dotted line to indicate where inversion occurs. The orange box represents WWII.

Unemployment Rate as Alternative High-Yield Bond

The unemployment rate (UNRATE) is highly correlated to the interest rate policy and is a major barometer of the health of the economy. We observe that, after WWII, the unemployment rate can be thought of as a high-yield bond. That is, in a low interest rate environment, the lowest unemployment rate tends to be very low. In a high interest rate environment, even the best unemployment rate is still quite high. This makes sense. Corporations have to compare the cost of employment to the cost of the capital.

In addition, the unemployment rate is a clear indicator how the economy responds to the yield spread tightening and easing. We find that the unemployment rate since 1950 (post-WWII) can be regressed in the following three-factor model:

$$UNRATE(t) \sim \beta_0 + \beta_1 GS_{10}(t) + \beta_2 GSI(t) + \beta_3 \widehat{YS}(t) + \varepsilon. \quad (9)$$

The role of the $\beta_2 GSI(t)$ term can be seen as modifying the weight of the mean $\langle YS(t) \rangle$ inside

$\widehat{YS}(t)$ from 1 to $(1 - \beta_2/\beta_3)$. This regression is shown below:

```
> J <- which(dt$fraction >= 1950 & dt$fraction <= 2019)
> a <- lm(unrate ~ rate.gs10 + rate.gs10.modinv + rate.spread.norm, data=dj[J])
> summary(a)
```

Call:

```
lm(formula = unrate ~ rate.gs10 + rate.gs10.modinv + rate.spread.norm,
    data = dj[J])
```

Residuals:

Min	1Q	Median	3Q	Max
-1.9918	-0.8227	-0.0183	0.6844	3.2611

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	2.33592	0.26146	8.934	<2e-16 ***
rate.gs10	0.40549	0.02408	16.839	<2e-16 ***
rate.gs10.modinv	3.36908	0.39905	8.443	<2e-16 ***
rate.spread.norm	4.74594	0.16454	28.843	<2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.028 on 824 degrees of freedom
Multiple R-squared: 0.6084, Adjusted R-squared: 0.607
F-statistic: 426.8 on 3 and 824 DF, p-value: < 2.2e-16

All three factors are highly significant. The R^2 is above 60%, which is quite high¹. $\beta_0 \approx 2.34$ can be interpreted as a baseline unemployment rate. The remaining terms expand into a complicated nonlinear relation on $GS_{10}(t)$ such as

$$\begin{aligned} UNRATE(t) &\sim \beta_0 + \beta_1 GS_{10}(t) + \beta_3 \frac{YS(t) - (1 - \beta_2/\beta_3) \langle YS(t) \rangle}{GS_{10}(t)} + \varepsilon. \\ &\sim (\beta_0 + \beta_3) + \beta_1 GS_{10}(t) + (\beta_2 - \beta_3) \frac{\langle YS(t) \rangle}{GS_{10}(t)} - \beta_3 \frac{GS_{3m}(t)}{GS_{10}(t)} + \varepsilon \end{aligned} \tag{10}$$

Figure 10 shows the result of this regression. $UNRATE(t)$ is the blue line and the regression is the red line. At each peak of a business cycle when $YS(t) \approx 0$, we have $GS_{3m}(t) \approx GS_{10}(t)$ and $\widehat{YS}(t) \approx -GSI(t)$. Then the lowest unemployment rate at that moment can be estimated by

$$\begin{aligned} UNRATE_{min}(t) &\equiv \beta_0 + \beta_1 GS_{10}(t) + (\beta_2 - \beta_3) \frac{\langle YS(t) \rangle}{GS_{10}(t)}. \\ &\approx 2.34 + 0.41 \times GS_{10}(t) - \frac{2.0}{GS_{10}(t)} \end{aligned} \tag{11}$$

Notice that $UNRATE_{min}(t)$ depends on $GS_{10}(t)$ alone. We observe in Figure 10 that $UNRATE_{min}(t)$ matches the lowest unemployment rate in each cycle peak quite well (with 1970 as a major exception). It provides a lower bound on the unemployment rate.

A cyclically increasing unemployment rate is the best confirmation of a bear market. There are unemployment spikes at the depths of some recessions, e.g. in 1974-75, in early 1980, and around 2010, that are not captured by the regression. These spikes, recorded by the residual ε , are what we call the “high-yield” elements in the unemployment rate.

In this business cycle, between 2013 and 2016, $UNRATE_{min}(t)$ went below 2% briefly when $GS_{10}(t)$ was below 1.85%. When the next yield spread will be inverted (in 2020?), if $GS_{10}(t)$ stays below 3%, the minimum unemployment rate can go as low as 3%.

¹Note that it would yield similar R^2 if the first factor is replaced with $\beta_1 GS_{3m}(t)$.

```

> plot(dj$fraction[J], dj$unrate[J], type="l", col="blue",
+       ylab="UNRATE", xlab="$t$", ylim=c(1.5,11),
+       main=sprintf("Unemployment rate as alternative high-yield bond"))
> lines(dj$fraction[J], predict(a, newdata=dj[J]), col="red")
> ac <- a$coefficients
> unrate.min <- ac[1]+ac[2]*dj$rate.gs10+(ac[3]-ac[4])*dj$rate.gs10.modinv
> lines(dj$fraction[J], ma(unrate.min,3)[J], col="black", lwd=2, lty=3)
> for (t in repo@yield.inversion) { abline(v=t, lty=2, col="red") }

```

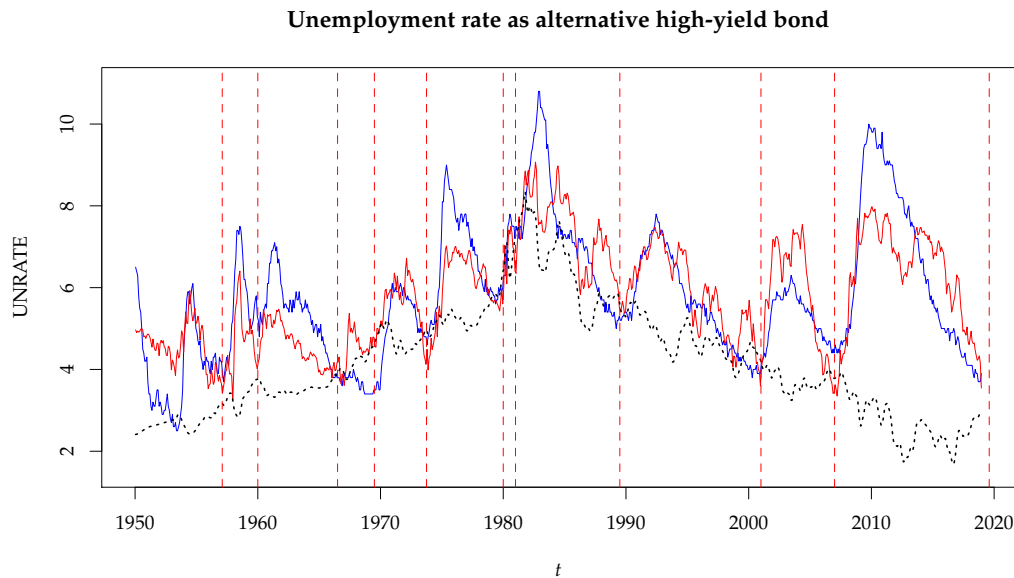


Figure 10: Unemployment rate $UNRATE(t)$ (the blue line) and its 3-factor regression to the bond market (the red line). The dotted vertical red lines mark the locations of yield spread inversion. The dotted black line is $UNRATE_{min}(t)$ (smoothed by 3-month moving average).

One-Year Change of Unemployment Rate

Just as important as observing the yield spread inversion, we also want to observe the inversion of the unemployment rate. The 1-year change (log-return) of the unemployment rate $r_{UNR}(t)$ is defined as

$$r_{UNR}(t) \equiv \log UNRATE(t) - \log UNRATE(t-1). \quad (12)$$

We find that $r_{UNR}(t)$ shows striking correlation with $-\widehat{YS}(t)$ and $Z_{20}(t)$, as shown in Figure 11. The combination of a weakening employment and the yield spread inversion appears to provide very convincing signal for a coming bear market. After the yield spread inversion, if $r_{UNR}(t)$ also turn negative, the recession is around the corner. The economy will continue to shed jobs until the yield spread is wide enough to provide corporations incentive to hire again. This process usually takes 12-24 months from past experiences.

```

> J <- which (dt$fraction >= 1950)
> plot(pred.20$fraction[J], pred.20$log.tri.smr[J], type="l", col="blue",
+       ylab="$Z_{20}(t)$", xlab="$t$", ylim=c(-0.6,0.6),
+       main="Z(t) and YoY log-change of UNRATE")
> lines(dj$fraction[J], -ma(dj$unrate.logr.1,3)[J], type="l", col="red", lwd=2)
> abline(h=0, col="blue", lty=2)
> lines(dj$fraction[J], ma(-dj$rate.spread.norm,3)[J], type="l", col="gray50")
> for (t in repo@yield.inversion) { abline(v=t, lty=2, col="gray50") }

```

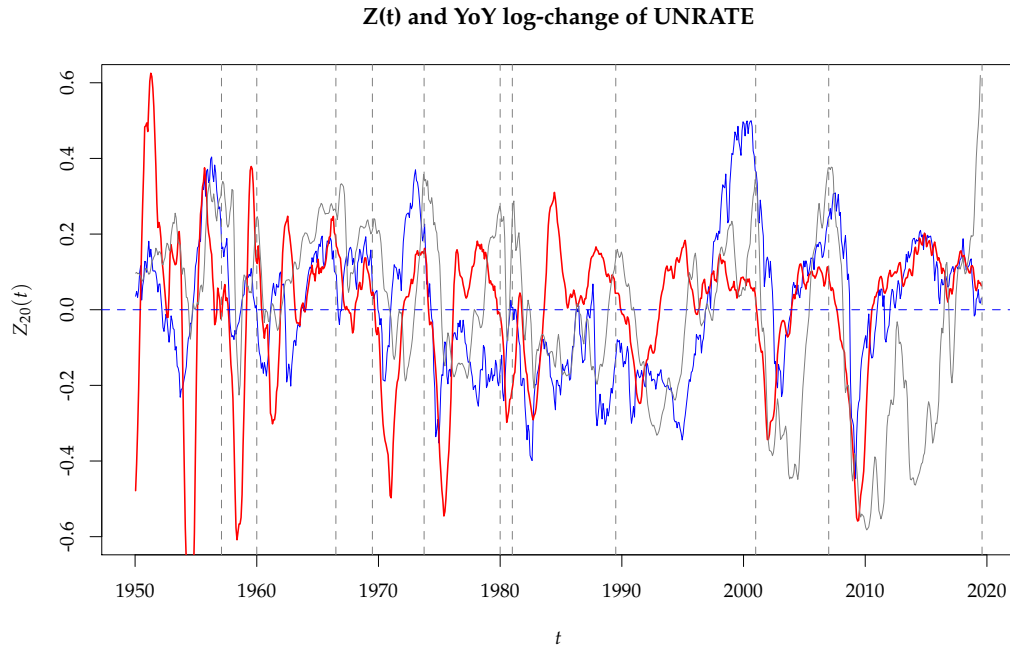


Figure 11: The reversal of $Z_{20}(t)$ (the blue line), $r_{UNR}(t)$ (the red line) and $-\widehat{YS}(t)$ (the gray line). When $r_{UNR}(t)$ turns negative and $\widehat{YS}(t)$ undergoes a yield spread inversion, it is a clear indicator that $Z_{20}(t)$ has entered a bear market.

Inflation

Lastly, in some circumstances, FED raises short-term rate in order to combat the rising inflation. Let $\Delta\text{CPI}_{f_1}(t)$ be the forward 1-year inflation rate $\text{CPI}_{f_1}(t)$ in excess of $GS_{10}(t)$. They are defined as

$$\begin{aligned} \text{CPI}_{f_1}(t) &\equiv \log \text{CPI}(t+1) - \log \text{CPI}(t); \\ \Delta\text{CPI}_{f_1}(t) &\equiv \text{CPI}_{f_1}(t) - GS_{10}(t). \end{aligned} \quad (13)$$

We find that $\Delta\text{CPI}_{f_1}(t)$ is correlated with the yield spread during some periods. This is illustrated in Figure 12. The policy of combating inflation is quite obvious in the 1970's. $\Delta\text{CPI}_{f_1}(t)$ moves in tandem with $-2(YS(t) - \langle YS(t) \rangle)$. After 1980, $\Delta\text{CPI}_{f_1}(t)$ is mostly negative, there has been a fairly large premium between $GS_{10}(t)$ and one-year inflation. Thus such policy goal is much less clear. After 2000, FED was able to maintain $\Delta\text{CPI}_{f_1}(t)$ barely below zero. Only recently, such premium is disappearing gradually, $\Delta\text{CPI}_{f_1}(t)$ is inching up above zero. And FED is indeed raising the short-term rate to combat it.

```

> plot(dj$fraction[J], (dj$cpilogr.f1*100-dj$rate.gs10)[J],
+      type="l", col="blue", ylab="$\\Delta\\mathrm{CPI}_{f1}(t)$", xlab="$t$",
+      ylim=c(-10, 10),
+      main=sprintf("YoY forward inflation vs yield spread"))
> lines(dj$fraction[J], (-2*(dj$rate.spread-ju@rate.spread.mean))[J], col="red")
> abline(h=0, lty=2, col="blue")
> abline(h=ju@rate.spread.mean*2, lty=2, col="blue")
> for (t in repo@yield.inversion) { abline(v=t, lty=2, col="red") }

```

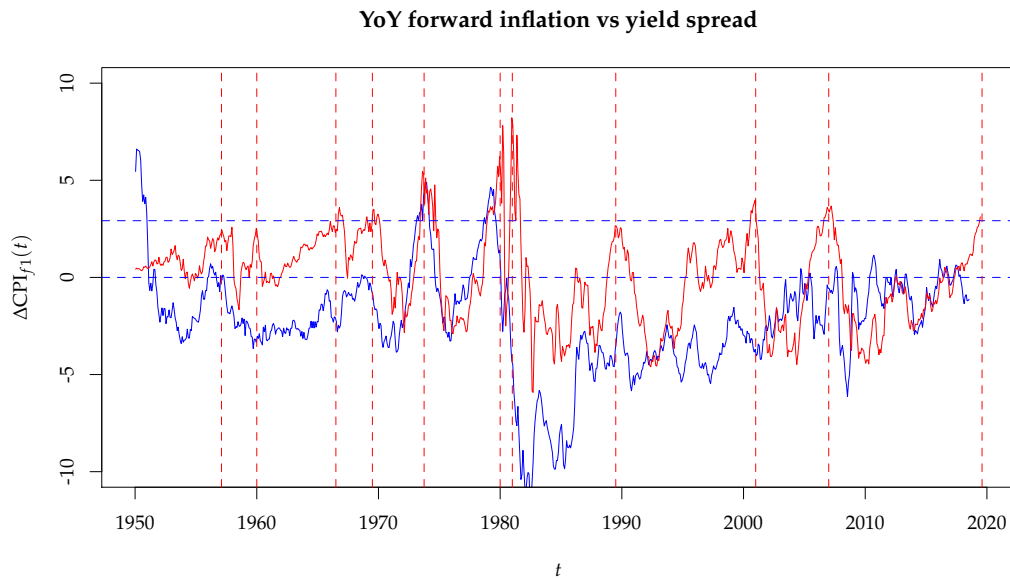


Figure 12: The 1-year forward inflation rate in excess of GS10: $\Delta\text{CPI}_{f_1}(t)$ (blue line). The red line is $-2(YS(t) - \langle YS(t) \rangle)$ indicating the strength of yield spread needed to suppress the runaway inflation. The top dotted blue line is the constant $2\langle YS(t) \rangle$, showing where the yield spread becomes inverted.

Discussion

We've shown that yield spread inversion is likely a major factor that causes the stock market to turn south. It can cause the stock market to drop 40-50% in 12-24 months. It also has a dire social consequence of causing the unemployment rate to spike. FED must invoke the inversion with extreme care.

We find that the best interpretation of FED's motivation to invoke a yield spread inversion is to cure an asset bubble. In 1970's, it was for the rising inflation, especially in gold and oil. This was consummated in 1980 with an ultimate peak of inflation, GS10, and the commodity bubble. In 1990, it was for the junk bond bubble and the Japanese stock market. Because the level of SPX was fairly subdued in terms of $Z_{20}(t)$ during this period, it didn't cause a big drop in SPX. In 2000, it was for the dot-com mania. In 2007, it was for the sub-prime debt bubble.

Thus in order to figure out the next bear market in $Z_{20}(t)$, one will need to watch a few things:

1. $YS(t)$: When will the yield spread become negative?
 - <https://fred.stlouisfed.org/series/T10Y3M>
2. $UNRATE(t)$ and $r_{UNR}(t)$: When will the unemployment rate begin to rise? When will the 1-year change turn negative?
 - <https://fred.stlouisfed.org/series/UNRATE>
3. $\Delta\text{CPI}_{f_1}(t)$: Is the 1-year inflation rate expected to exceed the 10-year Treasury yield?
4. Ultimately, is there an asset bubble that makes FED uncomfortable? In SPX? In Nasdaq? In the bond market? In any major foreign stock market?

Acknowledgement

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[p3]